Portfolio Optimization in *QuanTek*

(Revised July 22, 2017)

Robert Murray, Ph.D.
Omicron Research Institute

3.0 Contents

3.1 Need for Portfolio Optimization

The goal of the ***QuanTek*** program is to **maximize returns** and **minimize risk** in the portfolio. This is accomplished by a modification of the classical **Markowitz** method of portfolio optimization [SAB]. This method uses quantities that are estimated or measured from the price data of the securities in the portfolio, namely the **expected future return** and the **standard deviation** or **risk**. The **expected future return** is a quantity that must be estimated from some kind of **Price Projection**, which in ***QuanTek*** is provided by means of a **Linear Prediction** filter. The **standard deviation** is measured by taking a long-term average of the **average absolute deviation** and multiplying by a certain factor to convert it (assuming the Gaussian distribution) into the equivalent **standard deviation**. In a future version of ***QuanTek***, we hope to utilize the **Linear Prediction** method to estimate a time-varying future standard deviation, implementing a form of a **GARCH** (Generalized AutoRegressive Conditional Heteroskedasticity) [G] model.

In ***QuanTek*** the portfolio is optimized for a given **trading time horizon**, and the **future returns** are estimated for each security for this time horizon. As just stated, the **standard deviation** of each security is also measured, taking a long-term average of past returns. A third quantity that is measured is the **correlation** of the returns between all the securities, forming a (symmetric) **correlation matrix**. From the standard deviation, the square is taken to obtain the **variance** for each security. Then from the variance of each security and the correlation matrix, the correlation matrix is multiplied by the variances to obtain the **covariance matrix**. The quantities in the **portfolio optimization** calculation, for a portfolio of M securities, are the M **expected future returns** r_m and the M by M **covariance matrix** σ_{mn}^2 (where the indices run from 1 to M).

Given the fraction f_m of each security in the portfolio, the **portfolio expected return** $\bar{r}$ is the average over all the securities in the portfolio of the expected returns r_m:

$$\bar{r} = \sum_{m=1}^{M} f_m r_m$$

On the other hand, the total **variance** of the portfolio is given in terms of the **covariance matrix** σ_{mn}^2 by an average over all pairs of securities:

$$\bar{\sigma}^2 = \sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n$$

In our application, we will be optimizing the portfolio utilizing the **logarithmic returns** rather than the actual returns. This means that the quantities r_m are the **logarithmic expected returns** while the quantities σ_{mn}^2 are the **covariances** between the **logarithmic past returns**. This makes sense, because the price returns are expected to obey (approximately) a log-normal distribution. To then calculate the expected return and variance of the overall portfolio, we must convert these quantities to the actual expected returns and covariances, since the logarithmic returns are not directly additive. To do this, we assume that the log returns and log variances are additive over time, so the corresponding cumulative returns and variances are multiplicative. Then the square root of the cumulative variance is taken to obtain the standard deviation.

An important consequence of this is the concept of **portfolio diversification**. From the formula for the portfolio variance we can see that it becomes smaller as the number of securities in the portfolio becomes larger, if the portfolio fractions are evenly distributed among the securities. For example, if the variance of each security $\sigma_{mm}^2 \equiv \sigma^2$ is the same and all the securities are uncorrelated, and the fraction of each security in the portfolio is also the same so that $f_m = 1/M$, then the portfolio variance becomes:

$$\bar{\sigma}^2 = \sum_{m=1}^{M} f_m^2 \sigma_{mm}^2 \equiv M\left(\frac{1}{M^2}\right)\sigma^2 \equiv \frac{\sigma^2}{M} \quad (\text{M equal securities})$$

Hence we see that the total portfolio variance has been reduced by a factor *M* compared to the variance of each individual security. If the return of each security is the same, then the overall return is the same as the individual returns. Hence the **risk** of the **diversified portfolio** has been reduced while the **return** has remained the same.

This is the essence of the **portfolio optimization** problem – to find the optimal distribution of the portfolio weights in the portfolio, with *M* different securities having different **expected future returns** and **standard deviations**, in such a way as to maximize the overall **portfolio return** while minimizing the overall **portfolio risk**. This optimization depends on a choice of **risk aversion** or its opposite, **risk tolerance**, which determines the relative importance of the **returns** vs. the **risk**. An investor with low risk tolerance, or high risk aversion, will sacrifice returns in order to reduce risk, achieving the highest possible safety in the portfolio. An investor with a high risk tolerance, or low risk aversion, is more speculative and is willing to maintain more risk in order to achieve more returns. The **risk tolerance** enters as a parameter in the **portfolio optimization** problem, leading to a range of optimizations from minimum to maximum risk tolerance.

3.2 Outline of the Optimization Problem

The portfolio optimization problem consists of calculating the fraction f_m of each security in the portfolio that results in the optimal portfolio, **maximizing returns** and **minimizing risk**. The fraction f_m can be positive, corresponding to a long position, or negative, corresponding to a short position. The portfolio optimization is calculated by maximizing a certain function of the returns and risk, which I call the *Q*-function, subject to a constraint.

However, the nature of the constraint seems to make a big difference in the outcome. There are at least three possibilities. In the standard **Markowitz optimization**, the constraint is that the total value of the portfolio is held constant, with long positions counting as positive and short positions as negative. If we count these positions as fractions of the total equity, the sum of the positions equals a constant fraction, which we interpret as a margin percent:

$$\sum_{m=1}^{M} f_m = f_E \qquad (\text{Markowitz optimization})$$

However, this approach leads to a result that is not entirely satisfactory. This constraint is not very realistic. It leads to a result consisting of a **fixed portfolio** whose total value is f_E, which is the portfolio of least risk. The other part of the portfolio is called the **arbitrage portfolio** and has a total value of zero. In this portfolio the investor can invest in unlimited equal long and short positions and maintain a total value of zero, leading to unlimited risk. Normally this would never be allowed because the risk must be covered by the equity in the account.

Normally, in a margin account the short positions count toward the total equity in a positive sense, and must be covered by the equity just as are the long positions. Hence a more realistic constraint would be in terms of the sum of absolute values of the positions:

$$\sum_{m=1}^{M} |f_m| = f_E \qquad (\text{Standard optimization})$$

This is the realistic constraint, but it results in a very difficult problem to compute. The solution to this problem involves a method called **constrained quadratic optimization**, the solution to which was given by Markowitz [M]. (It is called constrained because the positions f_m are always constrained to be positive, if we do not allow short selling, as would be the case in a mutual fund portfolio, for example. Even if short selling is allowed, the absolute values in the constraint make the calculation much more difficult.) So for a practical portfolio calculation this is a difficult approach, but at least the constraint maintains the total absolute value of the investments in a margin account.

There is a third possibility, however. The mathematics becomes a lot simpler if a constraint involving a *sum of squares* can be used, analogous to the squared length of a space vector as the sum of squares of its components. Then the individual positions are unconstrained, allowing short selling, yet avoiding the troublesome absolute values. This method I call **unconstrained quadratic optimization**. So I propose the following constraint on the portfolio weights:

$$M \sum_{m=1}^{M} f_m f_m \equiv 1 \qquad (\text{Quadratic optimization})$$

This makes things easier because it does not involve the absolute values of the portfolio fractions, and also results in a better-defined optimization problem from the point of view of the *Q*-function. Notice that in all three optimization methods, if the total portfolio fraction of equity is set to $f_E = 1$, and you assume that all the security fractions are equal, then they are all equal to $f_m = 1/M$ of the total equity. Then the sum of all the fractions is unity and the portfolio is 100% invested. It turns out that when the positions are not all equal in the Quadratic optimization, the total of the absolute values f_M is less than unity. However, I view this as desirable, actually, because it results in a less risky portfolio when the equity is not evenly distributed and the diversification is not complete.

A side benefit, from the mathematical point of view, of the quadratic constraint is that if we ever want to diagonalize the covariance matrix, the constraint is invariant under an orthonormal transformation of the portfolio weights in the diagonalization. We might want to do this, for example, if we want to separate the portfolio into uncorrelated combinations or **factors** for use in a **factor model**.

Markowitz Mean-Variance Optimization

When the **Markowitz** portfolio optimization is performed, the result is a "convex cone" [G, p.132] of two portfolios (see below). The first portfolio is the one that minimizes risk, and it is called the **fixed portfolio**, because the total cost of this portfolio is fixed in the optimization. Given a certain amount of equity *E* available to the portfolio, we denote the (fixed) percentage allocated to the **fixed portfolio** by f_E. Then, for a portfolio with *M* securities, we have:

$$\sum_{m=1}^{M} f_m^{(\text{fix})} = f_E$$

The other portfolio that results from the portfolio optimization is called the **arbitrage portfolio**, because it turns out to have zero cost. Thus we have:

$$\sum_{m=1}^{M} f_m^{(\text{arb})} = 0$$

We define a (positive) coefficient γ which we call the **risk aversion**. In principle, as will be shown below, this coefficient could range from zero to infinity, but we will limit it to a finite range, to be explained below. This coefficient determines the amount of the arbitrage portfolio in the total portfolio. The total portfolio consists of the **fixed portfolio** plus the **arbitrage portfolio**. Thus the total portfolio may be written:

$$f_m = f_m^{(\mathrm{fix})} + f_m^{(\mathrm{arb})} \quad \Rightarrow \quad \sum_{m=1}^{M} f_m = f_E$$

Given that the total cost of the portfolio is f_E, the **margin leverage** f_M is given by the sum of the absolute values of all the positions:

$$\sum_{m=1}^{M} |f_m| = f_M \quad \text{with} \quad \sum_{m=1}^{M} f_m = f_E \qquad (f_E \le f_M)$$

Since the total cost f_E is fixed, the **margin leverage** f_M must be allowed to vary, depending on the chosen setting for the **risk aversion**. Typically, in a normal margin account at the present time, the maximum allowable **margin leverage** is 200%. (However, we find that when this **Markowitz** method is used, the actual margin leverage f_M can get very large, even while the total cost f_E is held at a fixed percentage.)

Unconstrained Quadratic Optimization

In the case of **unconstrained quadratic optimization**, to be explained further below, there is no division of the portfolio into two separate parts. The Q-function is optimized subject to the following quadratic constraint on the weights f_m:

$$M \sum_{m=1}^{M} f_m f_m \equiv 1 \quad \Rightarrow \quad \sum_{m=1}^{M} |f_m| \le 1$$

As stated previously, if all the weights f_m were equal, then we would get $f_m = 1/M$, and then the sum of the absolute values $|f_m|$ is again unity. However, if the f_m variables are not all equal, the sum of the absolute values $|f_m|$ is less than unity. Also, if we try to make the f_m variable as large as possible, by making one f_m much larger than all the others, we find that $f_m = 1/\sqrt{M}$ due to the constraint. In that case we also have $\sum_{m=1}^{M} |f_m| = 1/\sqrt{M}$. So we arrive at the modified constraint on the absolute values of the f_m variables, which is implicit in the above definition:

$$0 \le |f_m| \le 1/\sqrt{M} \quad \text{and} \quad 1/\sqrt{M} \le \sum_{m=1}^{M} |f_m| \le 1$$

This is actually a good definition for a mix of securities in a portfolio, because it helps ensure that the mix is balanced. Just as in the case of a mutual fund, in which the maximum equity that can be allocated to a single security is something like 5%, in the present case the maximum that can be allocated is $1/\sqrt{M}$, which for 25 securities (a rather large portfolio) will be 20%. However, if that were the case, then that single security would be the entire portfolio, so the **margin leverage** of the portfolio would also be 20%.

However, we would still like to be able to fix the amount of **margin leverage** f_M in the portfolio. We can do this by renormalizing the above weights and defining the sum of the renormalized weights $\hat{f}_m$ to be equal to the margin leverage:

$$\hat{f}_m \equiv \frac{f_m f_M}{\sum_{m=1}^{M} |f_m|} \quad \Rightarrow \quad \sum_{m=1}^{M} \left| \hat{f}_m \right| \equiv f_M \geq 0$$

We define the desired **margin leverage** f_M of the portfolio to be in the range $0 \leq f_M \leq 200\%$. Thus to get the renormalized weights, such that the sum of their absolute values equals the desired **margin leverage**, we just divide the computed weights by the sum of their absolute values and multiply by f_M. The weights were optimized using the quadratic constraint, but then renormalized to obey the absolute value constraint instead. The results are in general different than those obtained by optimizing directly using the absolute value constraint, but this method seems to actually be better as well as easier. For outlier securities such as those with extraordinarily high returns or low risk, the weights for these securities will be accentuated compared to the Markowitz method. (Optimizing in terms of mean-absolute values instead of root-mean-square values is more *robust*.) For well-diversified portfolios the results should be nearly the same.

3.3 Standard Mean-Variance Portfolio Optimization

Following Gouriéroux [G, p.130], the standard Mean-Variance optimization calculation due to **Markowitz** starts with the following Q-function:

$$Q(f) = \sum_{m=1}^{M} f_m r_m - \frac{\gamma}{2} \sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n \quad \text{with} \quad \sum_{m=1}^{M} f_m = f_E$$

The quantities f_m represent fractions of the total portfolio, and obey the constraint (last equation) that the total equity in the portfolio adds up to $f_E E$, where E is some fixed dollar amount of money, and f_E is the fraction of this equity. This cost function is then to be minimized by varying the weights f_m.

Gouriéroux introduces a Lagrange multiplier λ and writes the Q-function as follows:

$$Q(f;\lambda) = \sum_{m=1}^{M} f_m r_m - \frac{\gamma}{2}\sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n - \lambda\left(\sum_{m=1}^{M} f_m - f_E\right)$$

This is now minimized with respect to f_m and λ. This leads to two equations:

$$0 = r_m - \gamma\sum_{n=1}^{M}\sigma_{mn}^2 f_n - \lambda \quad \text{and} \quad \sum_{m=1}^{M} f_m = f_E$$

We may then multiply by the inverse covariance matrix and solve for the quantities f_n:

$$f_n = \frac{1}{\gamma}\sum_{m=1}^{M}\sigma_{nm}^{-2}\left[r_m - \lambda\right] \quad \text{with} \quad \sum_{m=1}^{M} f_m = f_E$$

Using the constraint we find:

$$f_E = \sum_{n=1}^{M} f_n = \frac{1}{\gamma}\left\{\sum_{n,m=1}^{M}\sigma_{nm}^{-2} r_m - \lambda\sum_{n,m=1}^{M}\sigma_{nm}^{-2}\right\}$$

Then we can solve for the value of the Lagrange multiplier:

$$\lambda = \frac{1}{\left\|\sigma^{-2}\right\|}\left[\left\|\sigma^{-2}r\right\| - \gamma f_E\right]$$

$$\text{where} \quad \left\|\sigma^{-2}\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2} \quad \text{and} \quad \left\|\sigma^{-2}r\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2} r_m$$

Finally the solution is:

$$\boxed{\begin{gathered} f_n = \frac{f_E}{\left\|\sigma^{-2}\right\|}\sum_{m=1}^{M}\sigma_{nm}^{-2} + \frac{1}{\gamma}\sum_{m=1}^{M}\sigma_{nm}^{-2}\left[r_m - \frac{\left\|\sigma^{-2}r\right\|}{\left\|\sigma^{-2}\right\|}\right] \\ \text{where} \quad \left\|\sigma^{-2}\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2} \quad \text{and} \quad \left\|\sigma^{-2}r\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2} r_m \end{gathered}}$$

We may write this as a sum of the two portfolios mentioned previously, the **fixed portfolio** and the **arbitrage portfolio**:

$$f_n = f_n^{(\text{fix})} + f_n^{(\text{arb})} \qquad f_n^{(\text{fix})} \equiv \frac{f_E}{\left\|\sigma^{-2}\right\|}\sum_{m=1}^{M}\sigma_{nm}^{-2}, \quad f_n^{(\text{arb})} = \frac{1}{\gamma}\sum_{m=1}^{M}\sigma_{nm}^{-2}\left[r_m - \frac{\left\|\sigma^{-2}r\right\|}{\left\|\sigma^{-2}\right\|}\right]$$

We see that the **fixed portfolio** is proportional to the total equity fraction f_E, corresponding to a riskless portfolio, while the **arbitrage portfolio** is proportional to $1/\gamma$, where γ is the **risk aversion**, and so corresponds to a risky portfolio. If the **risk aversion** parameter γ is taken to infinity, meaning the investor is maximally risk averse, the **arbitrage portfolio** goes to zero, and the portfolio is determined only by the **fixed portfolio** (in this limit). If the **risk aversion** parameter γ is taken to zero, meaning the investor has zero risk aversion, the portfolio is determined mainly by the **arbitrage portfolio**. The **fixed portfolio** depends on the inverse of the covariance matrix σ_{nm}^{-2}, and not on the estimated returns r_m at all. In fact, this portfolio is determined by the rows (summed over all columns) of the inverse covariance matrix. So, if the assets were totally uncorrelated, the weight of each asset would be proportional to the inverse of the variance of the asset, in a minimum risk portfolio. The **arbitrage portfolio** takes into account the expected returns, and hence involves risk.

If a sum is taken over all values of the index *n*, this gives the total (fractional) equity in the portfolio. It is easy to see that this sum is equal to f_E for the **fixed portfolio**, and zero for the **arbitrage portfolio**. (This is why it is called an *arbitrage* portfolio.) It is possible to increase the fractions f_n of equity in the **arbitrage portfolio** by arbitrary amounts, in long and short positions in proportion, in such a way that the total value is still zero. However, this would not be allowed in practice because of the constraint on the absolute values of the positions. In a real portfolio, if the risky arbitrage portfolio is increased, the riskless portfolio would be decreased so as to maintain the constant **margin leverage** of the portfolio, which is the sum of the *absolute values* of the fractions f_n. This is why the constraint of a constant *portfolio value* is an unrealistic constraint.

Mean-Variance Optimization with a Riskless Asset

If there is a riskless asset in the portfolio, such as cash or a money market fund, then we denote this asset with the zero index. Its risk-free return is r_0 and the fraction of it in the portfolio is f_0. Then the cost function to be minimized is [G, p.131]:

$$Q(f)=\sum_{m=0}^{M} f_m r_m - \frac{\gamma}{2}\sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n \quad \text{with} \quad \sum_{m=0}^{M} f_m = f_E$$

In this case we can simply eliminate f_0 using the constraint:

$$Q(f)=\left[\left(f_E-\sum_{m=1}^{M} f_m\right) r_0+\sum_{m=1}^{M} f_m r_m\right]-\frac{\gamma}{2}\sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n$$

This is now minimized with respect to f_m. This leads to the equations:

$$\frac{\partial Q(f)}{\partial f_m} \equiv 0=\left(r_m-r_0\right)-\gamma\sum_{n=1}^{M} f_n \sigma_{mn}^2$$

We again multiply by the inverse covariance matrix and solve for the quantities f_n:

$$f_n=\frac{1}{\gamma}\sum_{m=1}^{M} \sigma_{nm}^{-2}\left(r_m-r_0\right) \quad \text{with} \quad \sum_{m=0}^{M} f_m = f_E$$

This gives us the final solution:

$$\boxed{f_n=\frac{1}{\gamma}\sum_{m=1}^{M} \sigma_{nm}^{-2}\left(r_m-r_0\right) \quad \text{and} \quad f_0 = f_E-\frac{1}{\gamma}\sum_{n,m=1}^{M} \sigma_{nm}^{-2}\left(r_m-r_0\right)}$$

In this case, there is no clear separation between a fixed portfolio and an arbitrage portfolio. Rather, the riskless asset itself plays the role of the fixed portfolio. If $\gamma \to \infty$, then the entire equity fraction f_E is in the riskless asset f_0. Otherwise, the fraction in the riskless asset is reduced and that in the rest of the portfolio is increased. Eventually, as the constant γ becomes small, the riskless asset will be sold short in order to purchase more of the risky assets, which does not make much sense. So in this case there is a practical lower limit to the **risk aversion** parameter γ, namely the value for which the fraction of the riskless asset f_0 is reduced to zero. Once again, this is a consequence of the assumption that the sum of the equity fractions in the portfolio is equal to the total equity fraction f_E.

Statistical Properties of the Standard Mean-Variance Portfolio

Following Gouriéroux [G, p.133], we can calculate the expected return and variance of the optimal Mean-Variance portfolio, both without and with a riskless asset. For the case of no riskless asset, the total portfolio is a sum of two portfolios, one the **fixed portfolio** and the other the **arbitrage portfolio**. We start with the previously derived weights for these portfolios:

$$\boxed{f_n = f_n^{(\text{fix})} + f_n^{(\text{arb})} \qquad f_n^{(\text{fix})} \equiv \frac{f_E}{\left\|\sigma^{-2}\right\|}\sum_{m=1}^{M}\sigma_{nm}^{-2}, \quad f_n^{(\text{arb})} = \frac{1}{\gamma}\sum_{m=1}^{M}\sigma_{nm}^{-2}\left[r_m - \frac{\left\|\sigma^{-2}r\right\|}{\left\|\sigma^{-2}\right\|}\right]}$$

The expected returns of these two portfolios are then given by:

$$\mu\left(f_n^{(\text{fix})}\right) \equiv \sum_{n=1}^{M} r_n f_n^{(\text{fix})} = f_E\,\frac{\left\|\sigma^{-2}r\right\|}{\left\|\sigma^{-2}\right\|}, \quad \mu\left(f_n^{(\text{arb})}\right) = \sum_{n=1}^{M} r_n f_n^{(\text{arb})} = \frac{1}{\gamma}\left[\left\|r\sigma^{-2}r\right\| - \frac{\left\|\sigma^{-2}r\right\|^2}{\left\|\sigma^{-2}\right\|}\right]$$

$$\text{where}\quad \left\|\sigma^{-2}\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2}, \quad \left\|\sigma^{-2}r\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2}r_m, \quad \left\|r\sigma^{-2}r\right\| \equiv \sum_{n,m=1}^{M} r_n\sigma_{nm}^{-2}r_m$$

The expected return of the whole portfolio is then given by:

$$\mu\left(f_n\right) = \mu\left(f_n^{(\text{fix})} + f_n^{(\text{arb})}\right) = \mu\left(f_n^{(\text{fix})}\right) + \mu\left(f_n^{(\text{arb})}\right)$$

Similarly, the variance of the portfolio is given by:

$$\eta^2\left(f_n^{(\text{fix})}\right) \equiv \sum_{m,n=1}^{M} f_m^{(\text{fix})}\sigma_{mn}^2 f_n^{(\text{fix})} = f_E^2\,\frac{1}{\left\|\sigma^{-2}\right\|}, \quad \eta^2\left(f_n^{(\text{arb})}\right) = \sum_{n=1}^{M} f_m^{(\text{arb})}\sigma_{mn}^2 f_n^{(\text{arb})} = \frac{1}{\gamma^2}\left[\left\|r\sigma^{-2}r\right\| - \frac{\left\|\sigma^{-2}r\right\|^2}{\left\|\sigma^{-2}\right\|}\right]$$

$$\text{where}\quad \left\|\sigma^{-2}\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2}, \quad \left\|\sigma^{-2}r\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2}r_m, \quad \left\|r\sigma^{-2}r\right\| \equiv \sum_{n,m=1}^{M} r_n\sigma_{nm}^{-2}r_m$$

The variance of the whole portfolio is thus:

$$\eta^2\left(f_n\right) = \eta^2\left(f_n^{(\text{fix})} + f_n^{(\text{arb})}\right) = \eta^2\left(f_n^{(\text{fix})}\right) + \eta^2\left(f_n^{(\text{arb})}\right)$$

This follows because it can be verified that the covariance between the two portfolios is zero:

$$\sum_{m,n=1}^{M} f_m^{(\text{fix})}\sigma_{mn}^2 f_n^{(\text{arb})} = \sum_{m,n=1}^{M} f_m^{(\text{arb})}\sigma_{mn}^2 f_n^{(\text{fix})} = \frac{f_E}{\gamma}\left[\frac{\left\|\sigma^{-2}r\right\|}{\left\|\sigma^{-2}\right\|} - \frac{\left\|\sigma^{-2}\right\|}{\left\|\sigma^{-2}\right\|}\frac{\left\|\sigma^{-2}r\right\|}{\left\|\sigma^{-2}\right\|}\right] = 0$$

Thus it can be seen that the variance of the whole portfolio is always at least as great as the variance of the fixed portfolio alone, since both the terms are always positive.

Similarly, for the portfolio with a riskless asset, we start with the weights previously calculated:

$$\boxed{f_n = \frac{1}{\gamma}\sum_{m=1}^{M}\sigma_{nm}^{-2}\left(r_m - r_0\right) \quad \text{and} \quad f_0 = f_E - \frac{1}{\gamma}\sum_{n,m=1}^{M}\sigma_{nm}^{-2}\left(r_m - r_0\right)}$$

Then we find:

$$\boxed{\begin{aligned}\mu\left(f_n\right) &\equiv \sum_{n=0}^{M} r_n f_n = f_E r_0 + \frac{1}{\gamma}\sum_{n,m=1}^{M}\left(r_n - r_0\right)\sigma_{nm}^{-2}\left(r_m - r_0\right)\\ \eta^2\left(f_n\right) &\equiv \sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n = \frac{1}{\gamma^2}\sum_{n,m=1}^{M}\left(r_n - r_0\right)\sigma_{nm}^{-2}\left(r_m - r_0\right)\end{aligned}}$$

The variance of the portfolio is that of the risky assets alone, since the riskless asset has zero variance and zero covariance with the risky assets. It is interesting that in this portfolio (as in the portfolio with no riskless asset), the expected return of the risky assets is a constant times the variance of the risky assets. We note that the quantity,

$$S \equiv \sum_{n,m=1}^{M}\left(r_n - r_0\right)\sigma_{nm}^{-2}\left(r_m - r_0\right)$$

which is in general a function of time, is called **Sharpe's performance measure** for the portfolio [G, p.132,148]. We also have the **Sharpe performance coefficient**, defined for a single security as $\left(r_m - r_0\right)/\sigma_m$, where σ_m is the standard deviation of the returns [G, p.132].

The Capital Asset Pricing Model (CAPM)

Following Gouriéroux [G, p.183], and Sharpe (1964), Lintner (1965) and Mossin (1966), an equilibrium market condition can be added to the basic Mean-Variance portfolio optimization condition, to yield the **Capital Asset Pricing Model (CAPM)**. This condition says that the market as a whole constitutes a Mean-Variance optimal portfolio. Thus we can define weights of securities of the total market by the same optimization condition as for an individual portfolio (with riskless asset):

$$f_n^{(M)} = \frac{1}{\Gamma}\sum_{m=1}^{M}\sigma_{nm}^{-2}\left(r_m - r_0\right) \quad \text{and} \quad f_0^{(M)} = f_E^{(M)} - \frac{1}{\Gamma}\sum_{n,m=1}^{M}\sigma_{nm}^{-2}\left(r_m - r_0\right)$$

Here, (M) denotes the market quantities, and M denotes the number of securities in the whole market. The quantity Γ represents a **risk aversion** parameter for the market as a whole. In the present case, we must normalize the market portfolio so that the sum of the fractions $\tilde{f}_n^{(M)}$ is unity, with $f_E^{(M)}$ unity, and define Γ so that the fraction of the risk-free asset $f_0^{(M)}$ is zero. In other words, we specify the following:

$$f_E^{(M)} \equiv 1, \qquad \left\|\sigma^{-2}\left(r - r_0\right)\right\| \equiv \sum_{n,m=1}^{M}\sigma_{nm}^{-2}\left(r_m - r_0\right) \equiv \Gamma$$

This then yields:

$$\boxed{\begin{array}{c}\tilde{f}_n^{(M)} = \sum_{m=1}^{M} \sigma_{nm}^{-2}\left(r_m - r_0\right) \Big/ \left\|\sigma^{-2}\left(r - r_0\right)\right\| \quad \Rightarrow \quad \sum_{n=1}^{M} \tilde{f}_n^{(M)} \equiv 1 \\ \text{where} \quad \left\|\sigma^{-2}\left(r - r_0\right)\right\| \equiv \sum_{n,m=1}^{M} \sigma_{nm}^{-2}\left(r_m - r_0\right) \equiv \Gamma\end{array}}$$

The expected return and covariance matrix on the entire market is thus [G, p.184]:

$$\boxed{\begin{array}{l}\mu\left(\tilde{f}^{(M)}\right) \equiv \sum_{n=1}^{M} r_n \tilde{f}_n^{(M)} = r_0 + \sum_{n,m=1}^{M} \left(r_n - r_0\right) \sigma_{nm}^{-2}\left(r_m - r_0\right) \Big/ \left\|\sigma^{-2}\left(r - r_0\right)\right\| \\ \eta^2\left(\tilde{f}^{(M)}\right) \equiv \sum_{m,n=1}^{M} \tilde{f}_m^{(M)} \sigma_{mn}^{2} \tilde{f}_n^{(M)} = \sum_{n,m=1}^{M} \left(r_n - r_0\right) \sigma_{nm}^{-2}\left(r_m - r_0\right) \Big/ \left\|\sigma^{-2}\left(r - r_0\right)\right\|^2\end{array}}$$

Now we see from the above that we have the following relation between the expected return and variance on the entire market:

$$\mu\left(\tilde{f}^{(M)}\right) - r_0 = \eta^2\left(\tilde{f}^{(M)}\right)\left\|\sigma^{-2}\left(r - r_0\right)\right\|$$

We also need a covariance between an individual security and the market portfolio, which we denote by $\eta_m\left(\tilde{f}^{(M)}\right)$. Then this is given by:

$$\eta_m\left(\tilde{f}^{(M)}\right) \equiv \sum_{n=1}^{M} \sigma_{mn}^{2} \tilde{f}_n^{(M)} = \left(r_m - r_0\right) / \left\|\sigma^{-2}\left(r - r_0\right)\right\|$$

Thus, eliminating the constant quantity $\left\|\sigma^{-2}\left(r - r_0\right)\right\|$ (assumed positive) in the last two equations, we finally find:

$$\left(r_m - r_0\right) = \eta_m\left(\tilde{f}^{(M)}\right)\left\|\sigma^{-2}\left(r - r_0\right)\right\| = \eta_m\left(\tilde{f}^{(M)}\right) \frac{\left(\mu\left(\tilde{f}^{(M)}\right) - r_0\right)}{\eta^2\left(\tilde{f}^{(M)}\right)}$$

This is usually written in the following form:

$$\boxed{\left(r_m - r_0\right) = \frac{\eta_m\left(\tilde{f}^{(M)}\right)}{\eta^2\left(\tilde{f}^{(M)}\right)}\left(\mu\left(\tilde{f}^{(M)}\right) - r_0\right) \equiv \beta_m^{(M)}\left(\mu\left(\tilde{f}^{(M)}\right) - r_0\right)}$$

The quantity $\beta_m^{(M)}$ is called the *beta* of the security, with respect to the market portfolio (*M*). It expresses proportionality between the expected return of the security and that of the market portfolio in the CAPM. It is given by the covariance between the security and the market portfolio, divided by the variance of the market portfolio.

In practice, the CAPM can be viewed as a regression of the returns of each individual security on the hypothetical **market portfolio**. Another way of looking at it is that the CAPM is a **factor model** with a single factor – the market portfolio. It will be observed that the above equation has just the same form as the standard regression equation, for example when future returns are regressed on past returns of the same security. In the present case the future returns are regressed on the past returns of the market portfolio instead. The market portfolio can be simulated by an average such as the S&P 500. The returns of this market portfolio can then be viewed as what we have called a **technical indicator** – a function of past data that is supposed to be correlated with future returns. Thus, we can use a **Linear Prediction filter** to estimate the time-dependent covariance $\eta_m\left(\tilde{f}^{(M)}\right)$ between the future returns and the S&P 500 as the technical indicator.

3.4 Unconstrained Quadratic Optimization

In the unconstrained quadratic optimization, we start again with a *Q*-function defined as before, but use a quadratic constraint instead of a linear constraint:

$$Q(f) = \sum_{m=1}^{M} f_m r_m - \frac{\gamma}{2} \sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n \quad \text{with} \quad M \sum_{k=1}^{M} f_k f_k = 1$$

The constraint may be implemented by means of a Lagrange multiplier:

$$Q(f;\lambda) = \sum_{m=1}^{M} f_m r_m - \frac{\gamma}{2} \sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n - \frac{\lambda}{2}\left(M \sum_{k=1}^{M} f_k f_k - 1 \right)$$

The Q-function is then to be maximized with respect to the portfolio weights f_m and the Lagrange multiplier λ. This is a quadratic function, with the quadratic terms supplied both by the variance term and by the constraint. The linear term involving the returns contributes positively, and the quadratic variance term contributes negatively in proportion to the **risk aversion** coefficient γ. The presence of a quadratic term in this function is essential; otherwise the maximization problem has no solution.

This is to be compared with the usual Markowitz mean-variance optimization using a linear constraint:

$$Q(f) = \sum_{m=1}^{M} f_m r_m - \frac{\gamma}{2} \sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n \quad \text{with} \quad \sum_{k=1}^{M} f_k = f_E$$

In this case the Lagrange multiplier is implemented as follows:

$$Q(f;\lambda)=\sum_{m=1}^{M} f_m r_m - \frac{\gamma}{2}\sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n - \lambda\left(\sum_{k=1}^{M} f_k - f_E\right)$$

One problem with this linear constraint is that the problem has no solution in the case of zero **risk aversion**. In this case we take the volatility term to be zero and take as the *Q*-function:

$$Q_r\left(f\right)\equiv\sum_{m=1}^{M} f_m r_m - \lambda\left(\sum_{k=1}^{M} f_k - f_E\right)$$

This does not yield a solution due to the lack of a quadratic term, since the quadratic term in this case depends on the presence of the volatility term. Varying with respect to f_m yields:

$$\frac{\partial Q_r\left(f\right)}{\partial f_m}\equiv 0 = r_m - \lambda$$

This is merely a constraint on the returns (which are supposed to be given) rather than an equation for f_m. Thus this optimization procedure is ill-defined for the case of a trader who cares only about returns, not risk.

On the other hand, for the case of the quadratic constraint and zero **risk aversion** we use the following *Q*-function:

$$Q_r\left(f;\lambda\right)\equiv\sum_{m=1}^{M} f_m r_m - \frac{\lambda}{2}\left(M\sum_{k=1}^{M} f_k f_k - 1\right)$$

In this case, varying with respect to f_m yields:

$$\frac{\partial Q_r\left(f;\lambda\right)}{\partial f_m}\equiv 0 = r_m - \lambda M f_m \quad\Rightarrow\quad \lambda M f_m = r_m$$

In this case, we get a solution for f_m in which the fraction f_m is proportional to the expected return, which seems very reasonable. But this only came about due to the existence of the quadratic term in the Lagrange multiplier, not the quadratic variance term.

Minimizing with respect to the Lagrange multiplier λ gives us back the original constraint. We may now impose this constraint on the solution above:

$$\sum_{m=1}^{M} r_m r_m = \lambda^2 M^2 \sum_{m=1}^{M} f_m f_m = \lambda^2 M\left(M\sum_{m=1}^{M} f_m f_m\right) = \lambda^2 M$$

Let us now define the squared norm of the returns analogous to that used in the constraint:

$$|r|^2 \equiv M \sum_{m=1}^{M} r_m r_m$$

Thus we can solve for the Lagrange multiplier:

$$|r|^2 = \lambda^2 M^2 \quad \Rightarrow \quad |r| = \lambda M > 0$$

$$r_m = \lambda M f_m \quad \Rightarrow \quad f_m \equiv \hat{f}_m = \frac{r_m}{|r|} \equiv \hat{r}_m$$

Note that if all the returns are exactly zero, equivalent to the first term in the Q function absent, then there is no solution. This makes sense, because the return of the portfolio would be zero no matter what the portfolio weights.

Standard Markowitz *Q*-Function

If we go back to the original **Markowitz** Q-function, including a variance term to account for risk, and using the Lagrange multiplier λ to impose the constraint, we have:

$$Q_{r\sigma}\left(f;\lambda\right) \equiv \sum_{m=1}^{M} f_m r_m - \frac{\gamma}{2} \sum_{m,n=1}^{M} f_m \sigma_{mn}^2 f_n - \frac{\lambda}{2}\left(M \sum_{k=1}^{M} f_k f_k - 1 \right)$$

Minimizing with respect to all the variables f_m, we now find:

$$\frac{\partial Q_{r\sigma}\left(f;\lambda\right)}{\partial f_m} \equiv 0 = r_m - \gamma \sum_{n=1}^{M} \sigma_{mn}^2 f_n - \lambda M f_m$$

This now leads to the equation:

$$\boxed{\lambda M f_m = r_m - \gamma \sum_{n=1}^{M} \sigma_{mn}^2 f_n}$$

Imposing the constraint, we now find:

$$\sum_{m=1}^{M} \left(r_m - \gamma \sum_{n=1}^{M} \sigma_{mn}^2 f_n \right)\left(r_m - \gamma \sum_{n=1}^{M} \sigma_{mn}^2 f_n \right) = \lambda^2 M^2 \sum_{m=1}^{M} f_m f_m$$

$$\Rightarrow \quad \lambda^2 M^2 = M \sum_{m=1}^{M} \left(r_m - \gamma \sum_{n=1}^{M} \sigma_{mn}^2 f_n \right)\left(r_m - \gamma \sum_{n=1}^{M} \sigma_{mn}^2 f_n \right)$$

However, in the present case we are unable to find a simple algebraic solution for the Lagrange multiplier, except in the case $\gamma = 0$, which leads to the same solution as before. For $\gamma \neq 0$ there is no apparent way to eliminate the variables f_m on the right-side of the equation.

We can make progress by writing the equation above in matrix form (before imposing the constraint) and then inverting it to obtain f_m:

$$r_m = \sum_{n=1}^{M} \left[\lambda M \delta_{mn} + \gamma \sigma_{mn}^2 \right] f_n \equiv \sum_{n=1}^{M} K_{mn}(\lambda,\gamma)\, f_n \quad \Rightarrow \quad f_n = \sum_{m=1}^{M} K_{nm}^{-1}(\lambda,\gamma)\, r_m$$

If we now combine this with the constraint, we have an implicit equation for the Lagrange multiplier λ as a function of the parameter γ:

$$M \sum_{m=1}^{M} f_m f_m \equiv 1 = M \sum_{p,q=1}^{M} r_p K_{pq}^{-2}(\lambda,\gamma)\, r_q$$

We note that γ is defined to be an arbitrary non-negative constant, given as the **risk aversion**, while the Lagrange multiplier λ is determined (in principle) by γ using the constraint, as in the above implicit equation. The above equation is a constraint on the squared norm of the vector $\sum_{m=1}^{M} K_{nm}^{-1}(\lambda,\gamma)\, r_m$. We have already found the solution for λ when γ is zero. However, it is not clear how to obtain an explicit solution for λ when γ is non-zero. Given a fixed value of γ, we can, of course, find a solution for λ by an approximation procedure. The squared norm of the vector is computed, then the value of λ adjusted by successive approximations until the norm equals unity. So the problem should have a solution, at least within some range of the **risk aversion** parameter γ.

Limits on Constant Parameters

We insist that both the constants λ, γ, are non-negative, and at least one positive, in order that the matrix $K_{mn}(\lambda,\gamma)$ in brackets be positive definite and invertible. This should be the case provided γ is small enough, so that λ does not become negative, given the above equation for λ and γ:

$$\lambda M f_m = r_m - \gamma \sum_{n=1}^{M} \sigma_{mn}^2 f_n$$

We notice, in fact, that for a certain maximum value of γ, the value of λ should become zero:

$$\lambda \equiv 0 \quad \Rightarrow \quad r_m - \gamma_{(\max)} \sum_{n=1}^{M} \sigma_{mn}^2 f_n = 0$$

We take this maximum value of γ as the maximum **risk aversion**. Then, taking the value $\lambda = 0$, we may solve for the weights f_m for this case:

$$r_m = \sum_{n=1}^{M} \gamma_{(\max)} \sigma_{mn}^2 f_n \quad \Rightarrow \quad f_n = \frac{1}{\gamma_{(\max)}} \sum_{m=1}^{M} \sigma_{nm}^{-2} r_m \qquad (\lambda \equiv 0)$$

Thus we have:

$$M \sum_{n=1}^{M} f_n f_n = 1 = \frac{M}{\gamma_{(\max)}^2} \sum_{n,m=1}^{M} r_n \sigma_{nm}^{-4} r_m \quad \Rightarrow \quad \gamma_{(\max)}^2 = M \sum_{n,m=1}^{M} r_n \sigma_{nm}^{-4} r_m$$

Therefore we have:

$$\boxed{\gamma_{(\max)} = \sqrt{M \sum_{n,m=1}^{M} r_n \sigma_{nm}^{-4} r_m} \qquad (\lambda \equiv 0)}$$

Thus will be the maximum value of γ, and the minimum value will be zero.

On the other hand, the maximum value of λ will be that given in the previous section, which occurs when the value of γ is zero:

$$r_m = \lambda_{(\max)} M f_m \quad \Rightarrow \quad \lambda_{(\max)}^2 M^2 = |r|^2 \equiv M \sum_{m=1}^{M} r_m r_m$$

In that case, we solved for λ and found the value:

$$\boxed{\lambda_{(\max)} = \frac{1}{M} \sqrt{M \sum_{m=1}^{M} r_m r_m} \qquad (\gamma \equiv 0)}$$

Thus we see that as the **risk aversion** constant γ varies from zero to its maximum, the Lagrange multiplier λ varies from its maximum to zero. The zero lower limit of the **risk aversion** is mandated so that the investor will always be *risk-averse* instead of *risk-seeking*, whereas the upper limit of the **risk aversion** is somewhat arbitrary and is set to make sure the problem remains well-defined mathematically. (Theoretically the investor could have infinite **risk aversion**, meaning zero **risk tolerance**. In the present approach this would potentially result in the problem becoming ill-posed.)

Approximate Solution to the Optimization Problem

Now let is divide both sides of the above equation by $\lambda_{(\max)} M = |r|$:

$$r_m = \sum_{n=1}^{M} \left[\lambda M \delta_{mn} + \gamma \sigma_{mn}^2 \right] f_n \quad \Rightarrow \quad \hat{r}_m \equiv \frac{r_m}{|r|} = \sum_{n=1}^{M} \left[\frac{\lambda}{\lambda_{(\max)}} \delta_{mn} + \frac{\gamma}{|r|} \sigma_{mn}^2 \right] f_n$$

Using the above we recall:

$$\boxed{\lambda_{(\max)} = \frac{1}{M}\sqrt{M\sum_{m=1}^{M} r_m r_m} \equiv \frac{|r|}{M} \qquad \gamma_{(\max)} = \sqrt{M\sum_{n,m=1}^{M} r_n \sigma_{nm}^{-4} r_m} \equiv \left|r\sigma^{-2}\right|}$$

Thus we may write:

$$\boxed{\hat{r}_m \equiv \frac{r_m}{|r|} = \sum_{n=1}^{M}\left[\frac{\lambda}{\lambda_{(\max)}}\delta_{mn} + \frac{\gamma}{\gamma_{(\max)}}\hat{\sigma}_{mn}^2\right] f_n \qquad \hat{\sigma}_{mn}^2 \equiv \frac{\sigma_{mn}^2}{\left(|r| / \left|r\sigma^{-2}\right|\right)}}$$

Now let us define the following constants:

$$\alpha \equiv \frac{\lambda}{\lambda_{(\max)}} \quad \left(0 \le \alpha \le 1\right), \qquad \beta \equiv \frac{\gamma}{\gamma_{(\max)}} \quad \left(0 \le \beta \le 1\right)$$

However, we also know that $\alpha = 0 \Leftrightarrow \beta = 1$ and $\alpha = 1 \Leftrightarrow \beta = 0$. Thus we can *approximately* define $\alpha(\gamma) \approx \left(1-\beta(\gamma)\right)$. (Note that the **risk aversion** γ is actually the independent variable.) This equation is not exact, but it is exact at the two end points. At all other points it can be regarded as an approximate solution for the Lagrange multiplier $\lambda(\gamma)$. Then we invert the definition and express $\beta \approx (1-\alpha)$. It is seen that the parameter α can be interpreted as the (normalized) **risk tolerance** parameter.

Now we can redefine the parameters in the matrix as follows:

$$\boxed{\hat{r}_m \equiv \frac{r_m}{|r|} \approx \sum_{n=1}^{M}\left[\alpha\delta_{mn} + (1-\alpha)\hat{\sigma}_{mn}^2\right] f_n \equiv \sum_{n=1}^{M} \hat{K}_{mn}(\alpha)\, f_n}$$

Now it is clear that we have the limiting values $0 \le \alpha \le 1$ for the range of the parameter α:

$$\boxed{\alpha = 1: \quad f_m = \hat{f}_m = \hat{r}_m \qquad \alpha = 0: \quad f_m = \hat{f}_m = \sum_{n=1}^{M} \hat{\sigma}_{mn}^{-2}\hat{r}_n}$$

The value $\alpha = 0$ corresponds to minimum risk tolerance, and yields a minimum risk portfolio, taking into account the covariance matrix in the denominator. The value $\alpha = 1$ corresponds to maximum risk tolerance, and yields a maximum risk portfolio in which each position is proportional to the expected return, and the covariance matrix is not taken into account.

In the above, we have been using the notation $\hat{f}_m$ to mean the weights f_m that obey the quadratic constraint. The solutions given above obey this constraint exactly at the two endpoints of the risk tolerance parameter, and approximately elsewhere. However, we may instead want to normalize the weights so that the sum of their absolute values adds up to the **margin leverage**

f_M , as defined previously. We then reformulate the general solution and redefine the weights $\hat{f}_m$ to mean those renormalized to a given fixed margin leverage. The general solution is given by:

$$\boxed{f_n \approx \sum_{m=1}^{M} \hat{K}_{nm}^{-1}(\alpha)\,\hat{r}_m \qquad \hat{f}_n(f) \equiv f_n f_M \Big/ \sum_{m=1}^{M} \left| f_m \right|}$$

Thus the problem is solved by (approximately) defining a range of the parameters corresponding to risk tolerance, inverting a positive definite matrix, and then imposing a different constraint afterward by renormalizing.

This solution yields portfolios of minimum and maximum risk, which are in intuitive accord with what we would expect. Note that the minimum-risk portfolio here coincides with the maximum-risk portfolio of the standard Markowitz mean-variance treatment, and the minimum-risk portfolio in that case did not even take into account the expected returns. In the present solution, the portfolio weights are always proportional to the expected returns, with varying weight given to the covariance matrix in the denominator according to the risk tolerance. So this seems to yield a much more reasonable result than the standard Markowitz method.

3.5 Conclusion

The portfolio optimization calculation is important in order to **maximize returns** and **minimize risk** for the portfolio as a whole. It should be emphasized that only with a well-balanced portfolio is it reasonable to expect to achieve an acceptably low level of **risk**. If stocks or other securities are traded individually, most likely the outcome will be a drastic increase in risk and very little increase in return. It is necessary to have the optimized portfolio so that the random fluctuations in the different securities will balance each other and average out to yield a reasonable degree of risk reduction. Then, the idea is that instead of trading in each security separately, trading is done within the portfolio as a whole by the act of **portfolio rebalancing**, maintaining an **optimal portfolio** at all times.

The idea of trading by portfolio rebalancing automatically implements the **buy-low, sell-high** trading strategy. As the price of a security increases, with a fixed number of shares, so does the weight of that security in the portfolio. This results in an unbalanced portfolio. To keep the portfolio balanced, some of the shares should be sold. However, if the expected return is accurate, the portfolio optimization will recommend an increased position as long as the expected return is positive. Then, when the position reaches a peak and the expected return starts to decline, the position will be greater than the optimal weight in the portfolio, and a sell signal will be given. The selling will be spread out over time, but in general will occur *when the price is high and decreasing*. Then, with decreasing price, the weight of the security in the portfolio decreases. Again the portfolio is unbalanced. At some point, the expected return will again become positive, and assuming it is accurate, the portfolio optimization will again recommend an increased position. When the position reaches a trough and expected return starts to increase, the position will be less than the optimal weight in the portfolio, and a buy signal will be given. Again, the buying will be spread out over time, but in general will occur *when the price is low and increasing*. Thus, in this approach, the buy and sell signals are generally given near the troughs and peaks of the prices, but they are spread out over time, which reduces risk of a loss and spreads out the returns more evenly. So maintaining an **optimal portfolio** is a good way to implement a trading strategy that **maximizes returns** while **minimizing risk**.

3.6 References

[BP] Jean-Philippe Bouchaud & Marc Potters,
Theory of Financial Risks: From Statistical Physics to Risk Management, 2nd ed.,
Cambridge University Press, Cambridge, UK (2000)

[FFK] Frank J. Fabozzi, Sergio M. Focardi, & Petter N. Kolm,
Financial Modeling of the Equity Market: From CAPM to Cointegration,
John Wiley & Sons, Hoboken, New Jersey (2006)

[G] Christian Gourieroux, ARCH Models and Financial Applications,
Springer-Verlag, New York (1997)

[LCBP] Laurent Laloux, Pierre Cizeau, Jean-Philippe Bouchaud, & Marc Potters,
“Noise Dressing of Financial Correlation Matrices”,
Physical Review Letters, Vol.83, No.7 (16 August 1999)

[M] Harry Markowitz, “The Optimization of a Quadratic Function
Subject to Linear Constraints”, *Naval Research Logistics Quarterly*,
Vol.3, No.1-2 (March-June) pp.111-133 (1956)

[SAB] W.F. Sharpe, G.J. Alexander, & J.V. Bailey, Investments, 5th ed.,
Prentice Hall, Englewood Cliffs, NJ (1995)